



**ON THE ORIGINS
OF
DIGITAL SIGNAL PROCESSING**

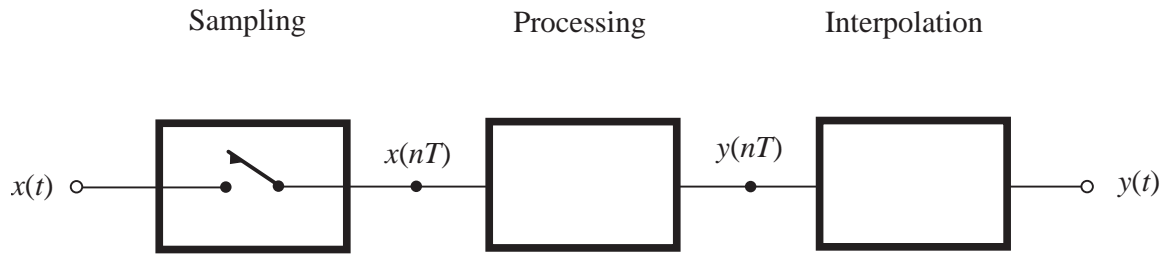
**Andreas Antoniou
University of Victoria
April 2003**



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WHAT IS DSP?

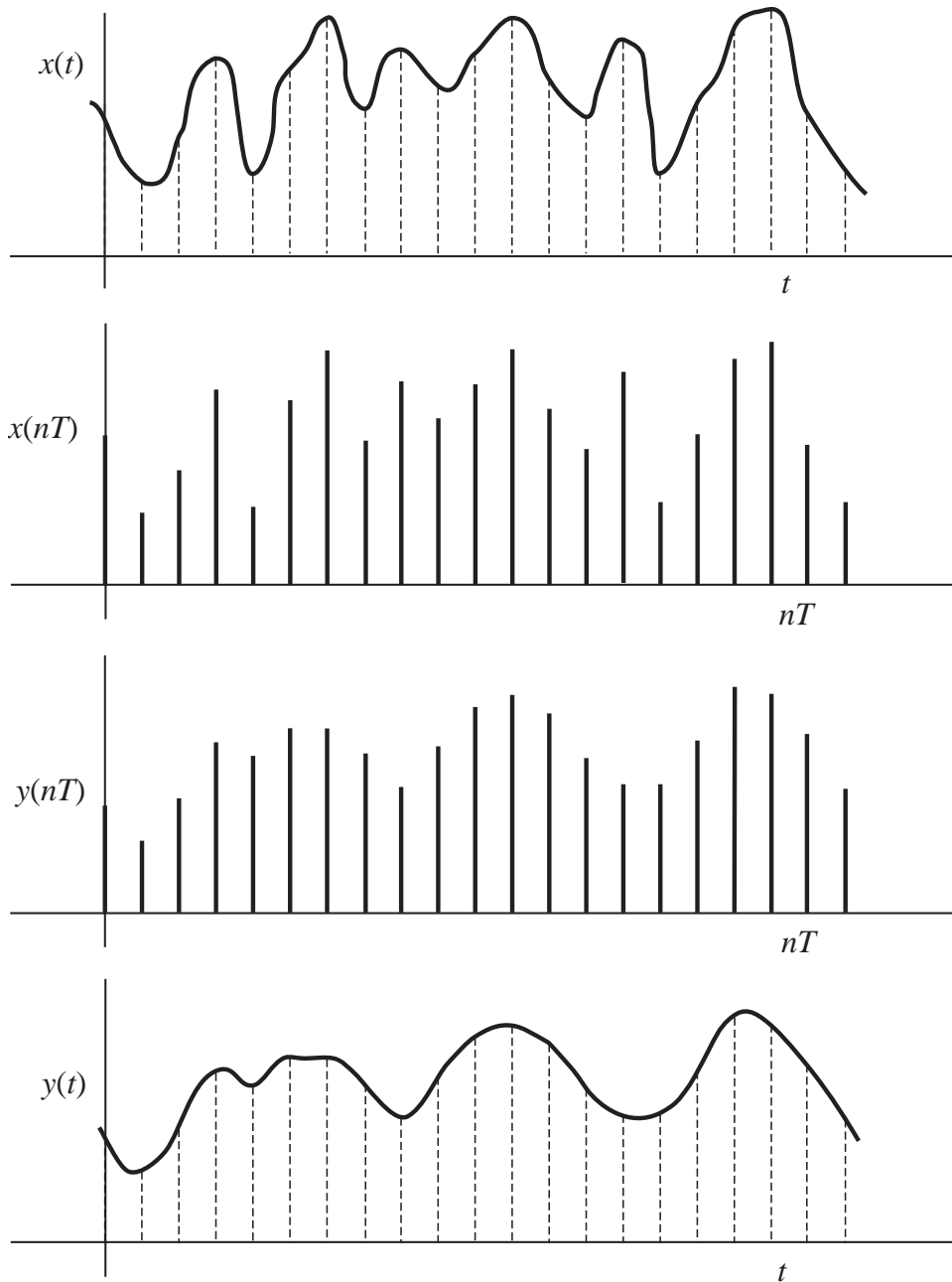


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WHAT IS DSP? Cont'd





What is DSP? *Cont'd*

To trace the origins of DSP, we must, therefore, trace the origins of the fundamental processes that make up DSP, namely,

- Sampling
- Processing
- Interpolation



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Archimedes of Syracuse

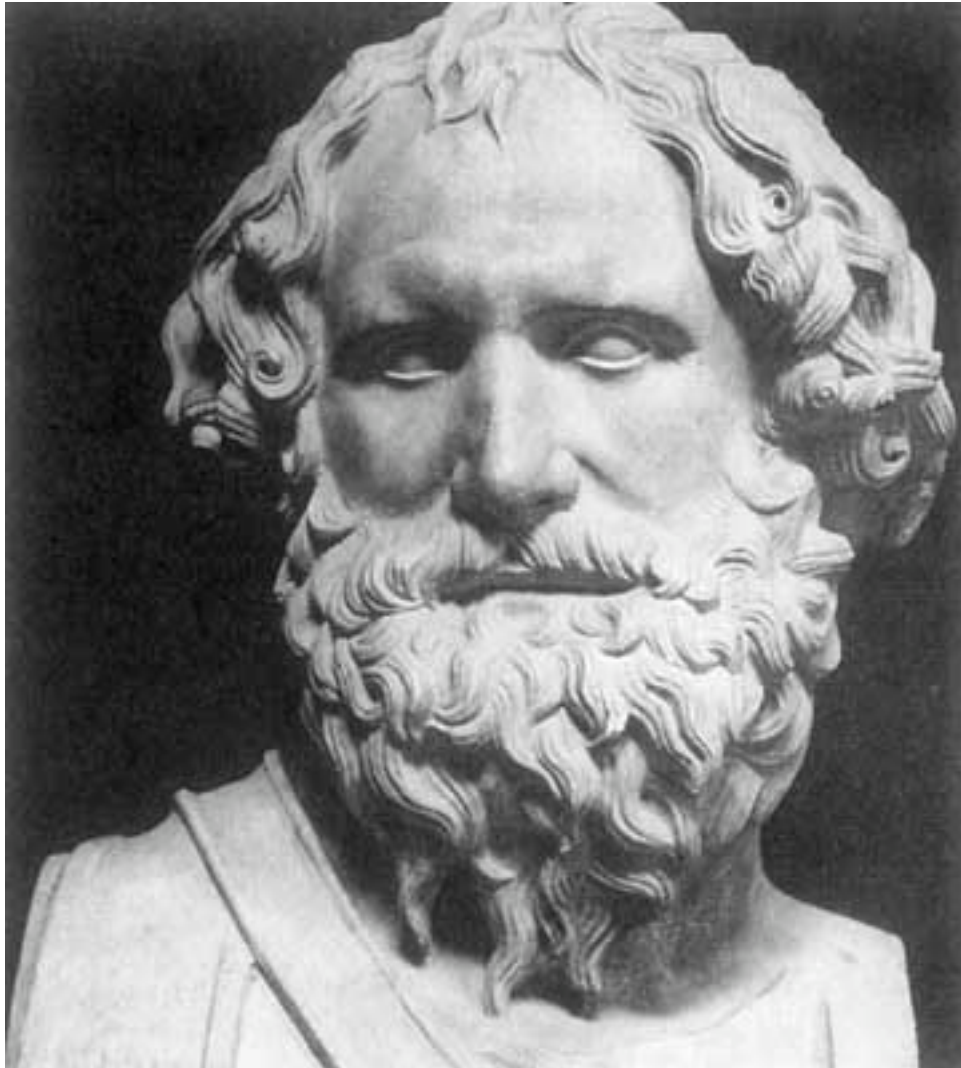
- The first occurrence of sampling and the first application of interpolation go back to Archimedes.
- Archimedes is most famous for the *the Archimedes principle* which gives the weight of a body immersed in a liquid.
- He developed fundamental theory for mechanics, wrote books on the subject, and is credited for many inventions, like the Archimedes' screw, and other things.
- He was also a great mathematician and the first person to give an algorithm for the calculation of π .
- He was born in Syracuse, Sicily, and lived during the period 287-212 BC.

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Archimedes of Syracuse

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Death of Archimedes

NOTE: These images and some of the biographical details mentioned here originate from the Web, see [1] on Foil 36.



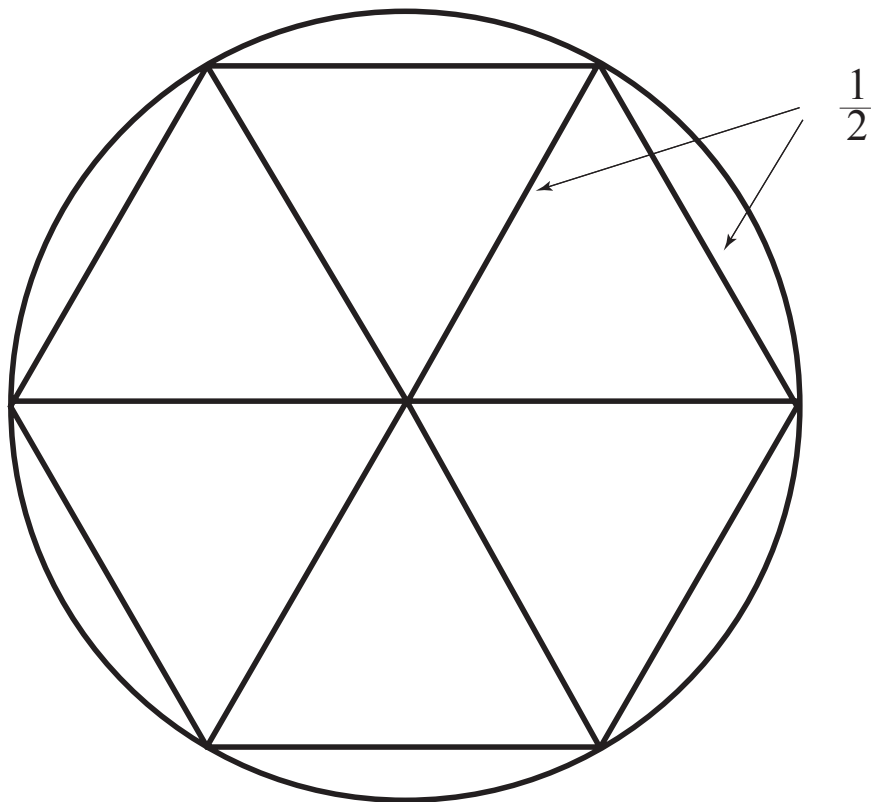
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Algorithm for π

- A lower bound for π can be readily obtained by inscribing a hexagon inside a circle of radius $\frac{1}{2}$.
- The regular hexagon can be broken down into 6 equilateral triangles; hence the perimeter of the hexagon, denoted as p_6 , is $6 \times \frac{1}{2} = 3$, i.e., $p_6 = 3$.
- The perimeter of the inscribed hexagon is obviously smaller than the circumference the circle, which is $2\pi \times radius = \pi$, i.e.,

$$3 < \pi$$



$$p_6 = 3$$

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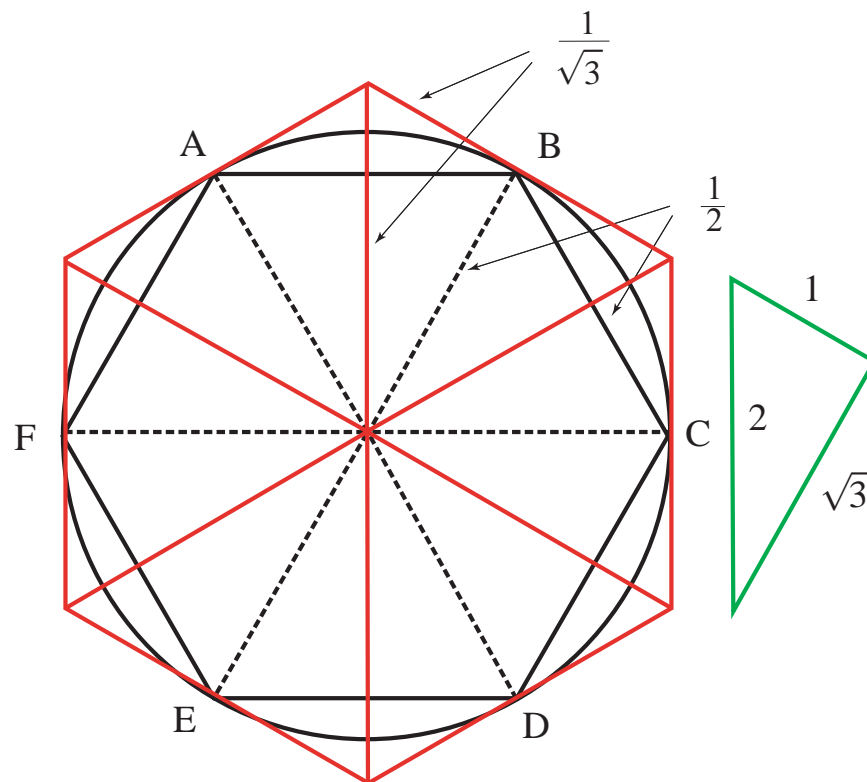
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Algorithm for π Cont'd

- An upper bound for π can be readily obtained by circumscribing a circle of radius $\frac{1}{2}$ by a hexagon.
- Draw tangents at points A, B, C, D, E, and F as shown.
- The perimeter of the larger hexagon is given by $P_6 = 6 \times 1/\sqrt{3} = 2\sqrt{3} = 3.4641$.
- The circumference of the circle, π , is smaller than the perimeter of the larger hexagon; hence we have

$$p_6 = 3 < \pi < 2\sqrt{3} = P_6$$



$$P_6 = 2\sqrt{3}$$

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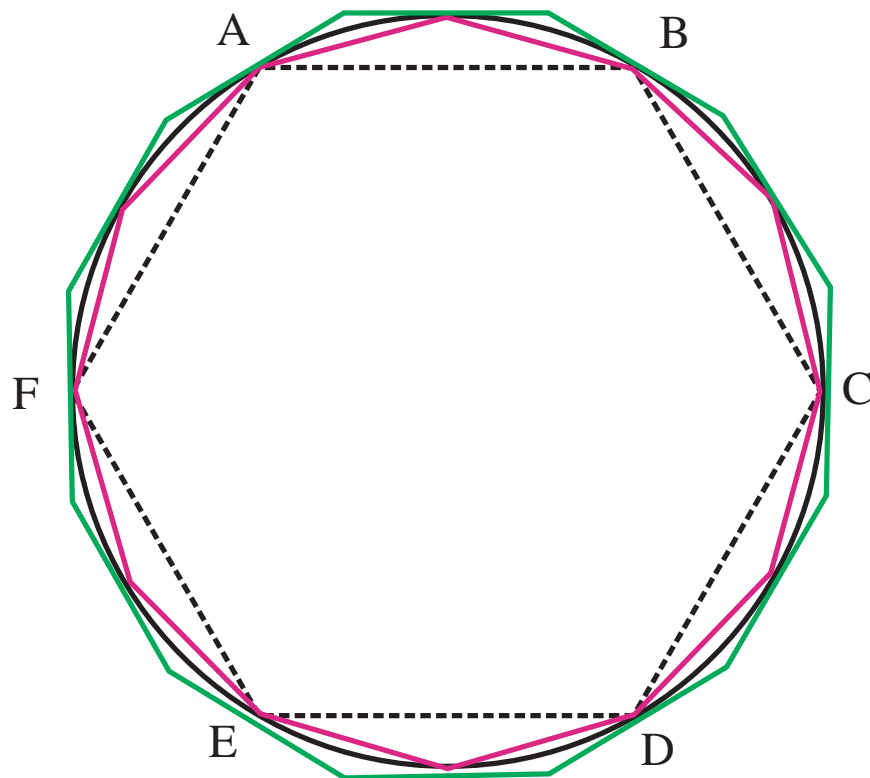
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Algorithm for π Cont'd

- Tighter lower and upper bounds on π can be readily obtained by using 12-sided regular polygons (dodecagons) instead of 6-sided ones, as shown below.
- The inside dodecagon is obtained by drawing straight lines that divide the arcs AB, BC, etc.
- The outside dodecagon is obtained by drawing tangents at the 12 vertices of the inside dodecagon.



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Algorithm for π Cont'd

- **Geometry will show that the perimeters of the larger and smaller dodecagons, namely, P_{12} , and p_{12} , are given by**

$$P_{12} = \frac{2p_6P_6}{p_6 + P_6} \quad \text{and} \quad p_{12} = \sqrt{p_6P_{12}}$$

respectively, or

$$P_{2 \times 6} = \frac{2p_6P_6}{p_6 + P_6} = \frac{2 \times 3 \times 3.4641}{3 + 3.4641} = 3.2154$$

and

$$p_{2 \times 6} = \sqrt{p_6P_{2 \times 6}} = \sqrt{3 \times 3.2154} = 3.1058$$

Therefore, we have

$$3 < 3.1058 < \pi < 3.2154 < 3.4641$$

or

$$p_6 < p_{2 \times 6} < \pi < P_{2 \times 6} < P_6$$

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Algorithm for π Cont'd

Archimedes found out that the same procedure can be repeated with 24-sided, 48-sided, and 96-sided regular polygons.

Furthermore, a recursive pair of formulas can be constructed that can be used to calculate the perimeters of successive outside and inside polygons, namely,

$$p_{2n} = \frac{2p_n P_n}{p_n + P_n} \quad \text{and} \quad P_{2n} = \sqrt{p_n P_{2n}}$$

respectively (see [2] for details).

Table 1 Bounds for π

No. of sides	Lower bound	Upper bound
6	3.0000	3.4641
12	3.1058	3.2154
24	3.1326	3.1597
48	3.1394	3.1461
96	3.1410	3.1427
⋮	⋮	⋮

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Algorithm for π Cont'd

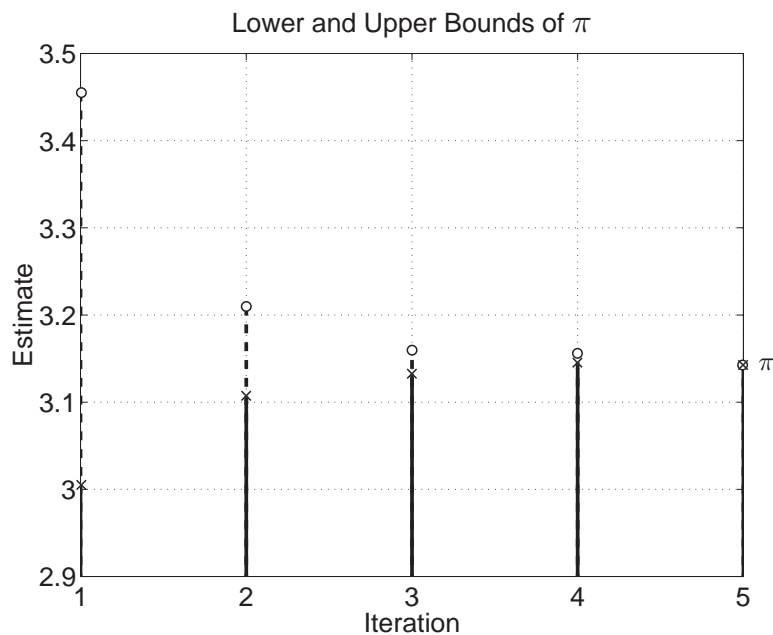
- For some reason, Archimedes' stopped on the 5th iteration which would involve 96-sides polygons.

The interpolated (or average) value of π on the 5th iteration is

$$\pi \approx \frac{1}{2}(3.1410 + 3.1427) = 3.1419$$

It entails an error of about 0.01%.

- The approximation $\frac{22}{7} = 3.1429$, which is the closest ratio of integers to the above interpolated value of π , is known as the **Archimedean value of π** .
- Archimedes' algorithm would yield π to a precision of 1 part 10^{10} in 17 iterations, which would entail the use of 393216-sided regular polygons.
- Archimedes was, in effect, the first to apply sampling and interpolation.



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INTERPOLATION

- Interest in interpolation resurfaced in Europe some 2000 years later during the middle ages while the scientists of the time were trying to fit curves to measured experimental data.

For example, in an attempt to characterize the orbits of planets and other celestial objects.

- Through induction and interpolation techniques not very different from those of Archimedes, John Wallis (1616-1703) showed that

$$\frac{\pi}{2} = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots}{1 \cdot 3^2 \cdot 5 \dots}$$

- He was an English mathematician who also served as chaplain of king Charles II.
- By the way, he was the first person to use the infinity symbol ∞ .

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INTERPOLATION *Cont'd*

- **James Gregory (1638-1675), extended the results of Archimedes on the circumference and area of the circle to the ellipse and hyperbola [2].**
- **He discovered the Taylor series in 1671, 44 years before Taylor published it.**
- **He was a Scot mathematician.**



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INTERPOLATION *Cont'd*

- Sir Isaac NEWTON (1642-1727) began his work by exploring the representation of functions by infinite series and came up with the binomial theorem.
- He then moved on to bigger and better things.



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INTERPOLATION *Cont'd*

- **Brook Taylor (1685-1731), a Cambridge graduate and an admirer of Newton, published the Taylor series in 1715 and made contributions on the solution of differential equations.**



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INTERPOLATION *Cont'd*

The interpolation process was explored by many since the time of Newton:

- **James Stirling (1692-1770)**, a friend of Newton, contributed to interpolation and added to the work of Newton on cubic curves.
- **Joseph-Louis Lagrange (1736-1813)** is claimed to be a French or Italian mathematician by the French or Italians. He was born in Turin but his father had French ancestry. He contributed to astronomy, number theory, and calculus.
- **Wilhelm Bessel (1784-1846)** left the gymnasium at 14 and joined an import-export company. While working for this company, he got interested in geography and later in astronomy. Eventually, he wrote an excellent paper on the trajectory of Halley's comet. This work was soon recognized by the scientific community of the time that led to Gauss recommending him for a doctorate. He dedicated the rest of his life to astronomy and mathematics.

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THE STIRLING INTERPOLATION FORMULA

If the values of $x(nT)$ are known at $0, T, 2T, \dots$, then the value of $x(nT + pT)$ for some value of p in the range $0 < p < 1$ can be determined as

$$\begin{aligned}
 x(nT + pT) = & \left[1 + \frac{p^2}{2!}\delta^2 + \frac{p^2(p^2 - 1)}{4!}\delta^4 + \dots \right] x(nT) \\
 & + \frac{p}{2}[\delta x(nT - \frac{1}{2}T) + \delta x(nT + \frac{1}{2}T)] \\
 & + \frac{p(p^2 - 1)}{2(3!)}[\delta^3 x(nT - \frac{1}{2}T) + \delta^3 x(nT + \frac{1}{2}T)] \\
 & + \frac{p(p^2 - 1)(p^2 - 2^2)}{2(5!)}[\delta^5 x(nT - \frac{1}{2}T) + \delta^5 x(nT + \frac{1}{2}T)] \\
 & + \dots
 \end{aligned}$$

where

$$\delta x(nT + \frac{1}{2}T) = x(nT + T) - x(nT)$$

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STIRLING FORMULA, ALTERNATIVE FORM

For $p = \frac{1}{2}$, eliminating the central differences gives

$$y(nT) = x(nT + \frac{1}{2}T) = \sum_{i=-3}^3 h(iT)x(nT - iT)$$

where coefficients $h(iT)$ are given in the following table.

Table 2
Coefficients $h(iT)$

i	$h(iT)$
-3	0.005859
-2	-0.062500
-1	0.498047
0	1.000000
1	-0.185547
2	0.046875
3	-0.005859

Evidently, Stirling's formula can be used to design an FIR (nonrecursive) discrete-time system that can perform interpolation.

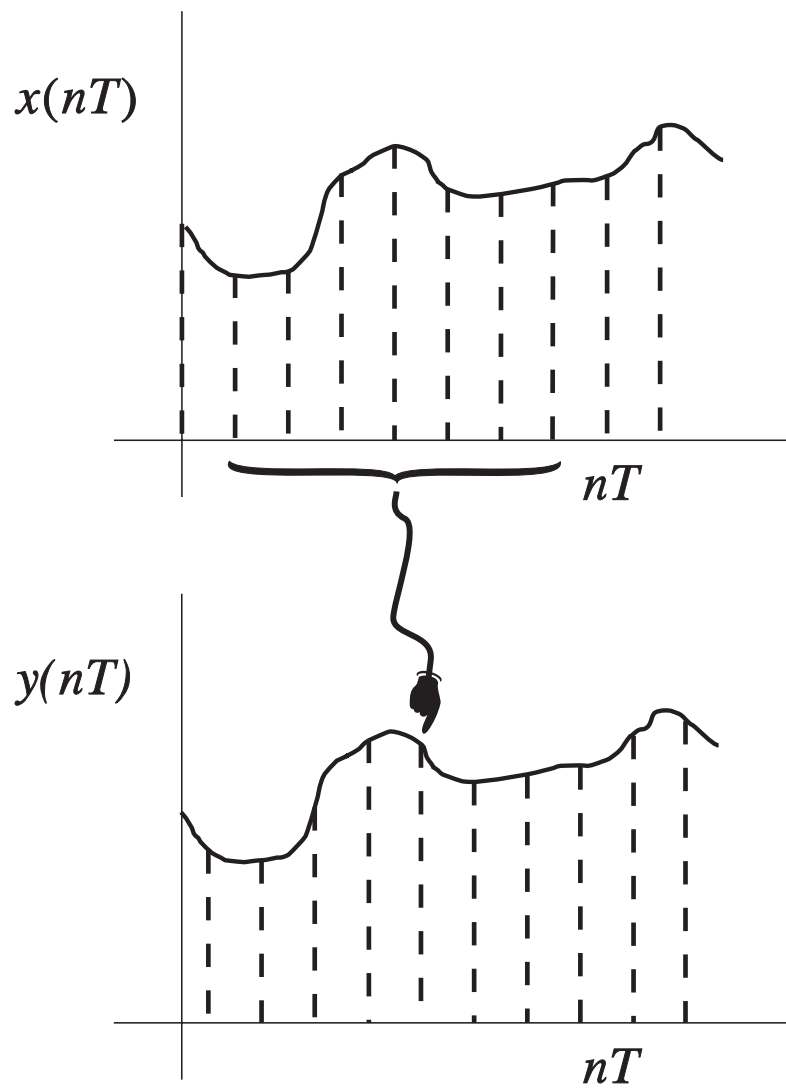
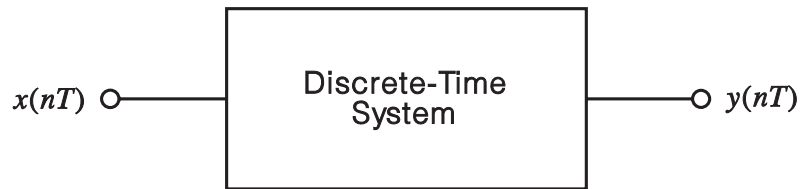
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STIRLING FORMULA *Cont'd*



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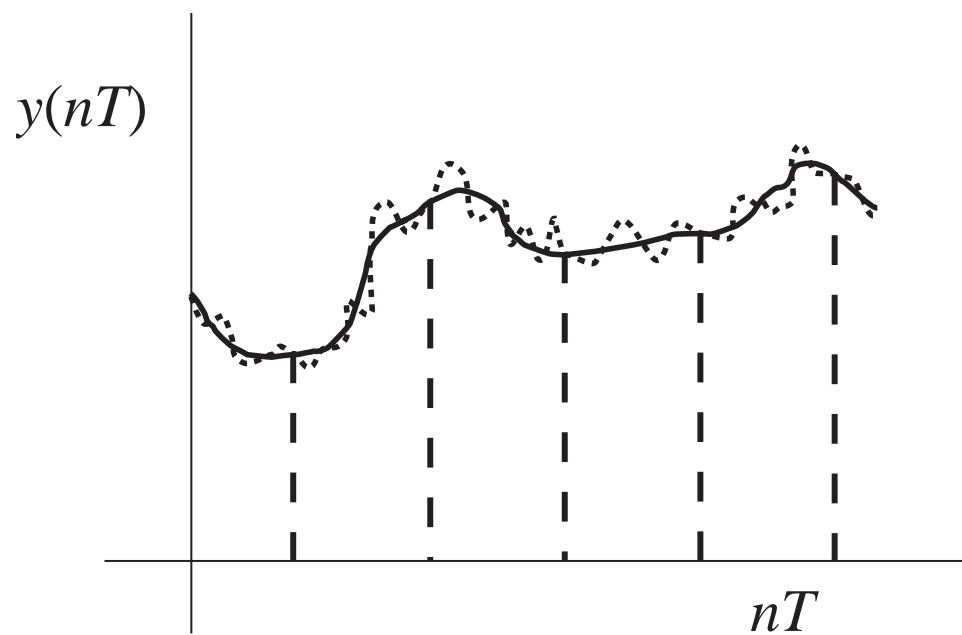
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STIRLING FORMULA *Cont'd*

- Interpolation is a process that will fit a smooth curve through a number of sample points.
- In effect, interpolation is akin to lowpass filtering.



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STIRLING FORMULA *Cont'd*

Our interpolation system can be represented by the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=-3}^3 h(iT)z^{-k}$$

Hence its frequency response, amplitude response, and phase response are given by

$$H(e^{j\omega T}) = \sum_{i=-3}^3 h(iT)e^{-jk\omega T},$$
$$M(\omega) = \left| \sum_{i=-3}^3 h(iT)e^{-jk\omega T} \right|, \quad \text{and}$$
$$\theta(\omega) = \arg \sum_{i=-3}^3 h(iT)e^{-jk\omega T}$$

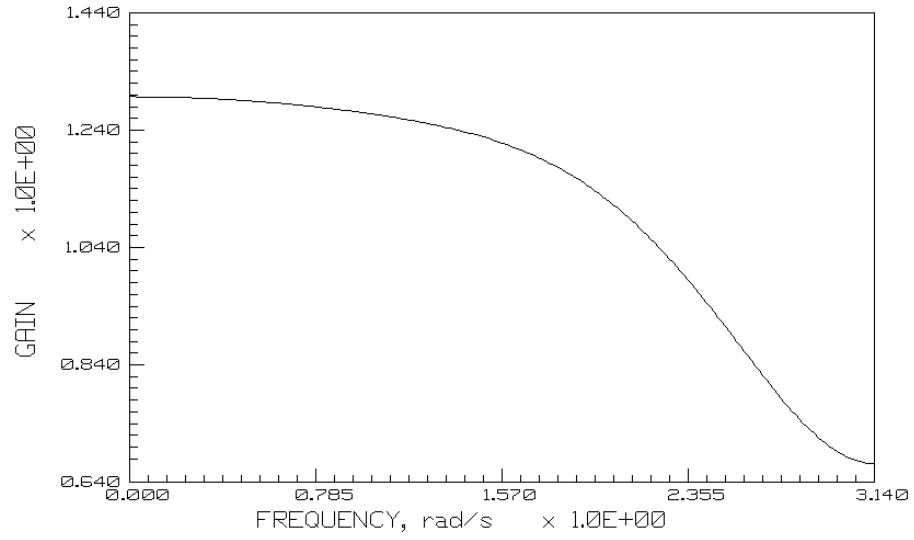
respectively.

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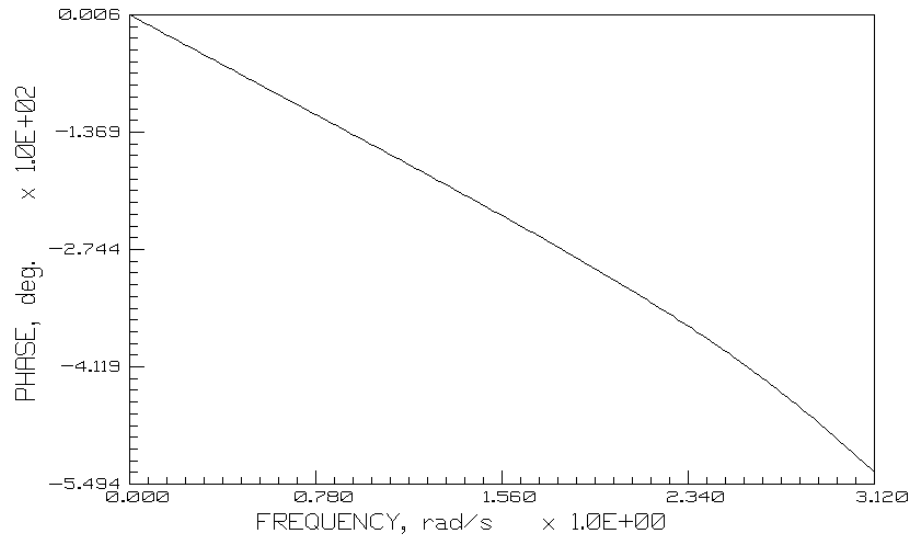
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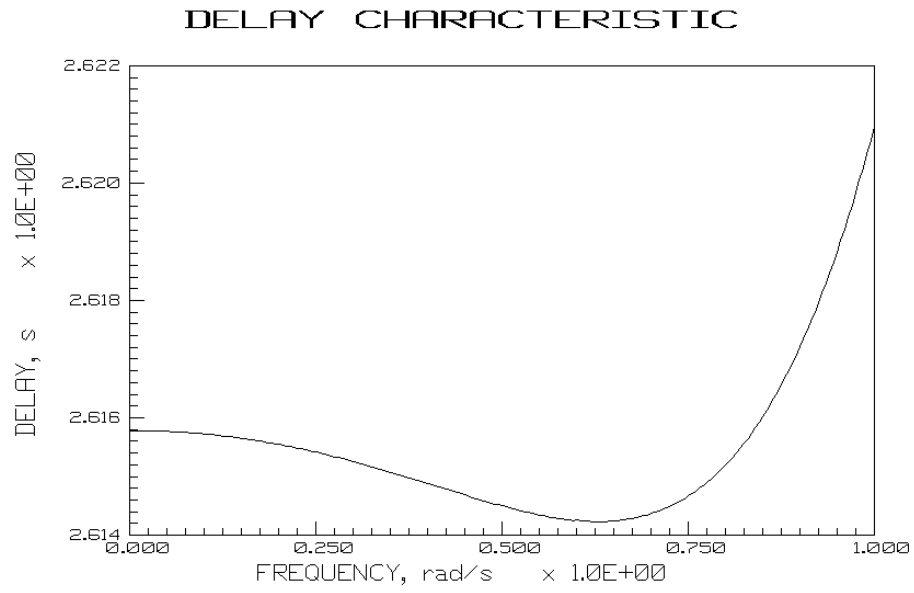


AMPLITUDE RESPONSE



PHASE RESPONSE





NOTE: Variation of group delay over the range 0 to 1 rad/s is 0.26 %.



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INTERPOLATION *Cont'd*

These transparencies have shown that interpolation formulas invented 250 years ago, actually represent what we refer to today as FIR lowpass filters.

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NUMERICAL TABLES FOR EVERYTHING

From the time of Newton and up to very recently, society relied heavily on numerical tables for

- **navigation**
- **banking**
- **astronomy**
- **science**
- **engineering**
- **etc.**

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NUMERICAL TABLES FOR EVERYTHING *Cont'd*

A table of a function is obviously a sampled version of the function.

In other words, even if we chose to ignore Archimedes attempt to find a value for π by obtaining different approximations for the value of the circumference of the circle, the sampling process has been around for centuries.



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THE CRISIS OF THE TABLES

- Typically a volume of tables contained numerous errata due to calculation, copying, and typesetting errors.
- A certain Dionysius Lardner wrote in 1834 that a random selection of 40 volumes of numerical tables chosen at random contained 3400 acknowledged errata and many more that were not acknowledged.
- An Englishman by the name of Charles Babbage (1791-1871) estimated that the British government had lost 2 to 3 million pounds as a direct consequence of numerical errors in published tables.

He was trying to get a grant from the government at that time to develop a machine that would calculate the numerical data and, in addition, produce the lead type that would print the tables, all without human intervention.

He was successful in getting grants and spent the rest of his life trying to design such a machine.

See [3] for an interesting article on Babbage.

- Charles Babbage lived in London near the present day Wax Museum not far from Sir Arthur Conan Doyle's residence on Baker Street.

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Charles Babbage

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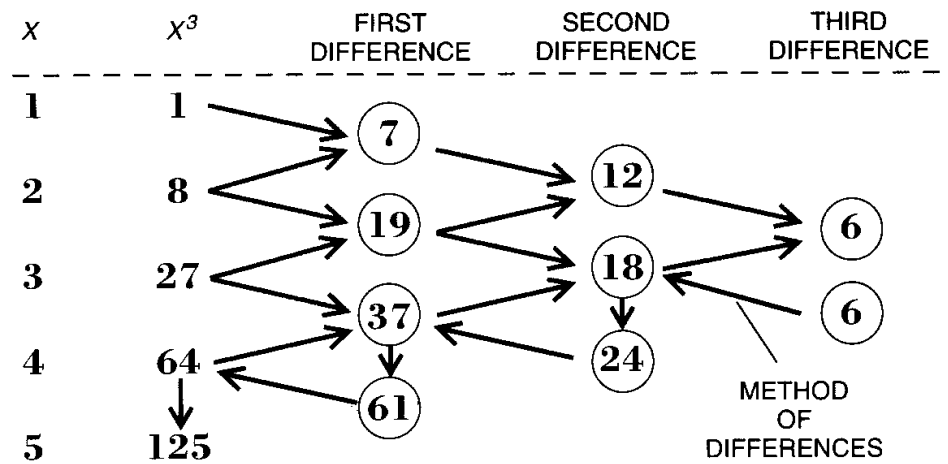
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BABBAGE'S BASIC IDEA

Difference table for $f(x) = x^3$



NOTES:

- The third difference is a constant for any third-order polynomial.
- For the same reason, the N th-order difference for an N th-order polynomial is a constant.

This is so for the same reason that the N th derivative of an N th-order polynomial is a constant.

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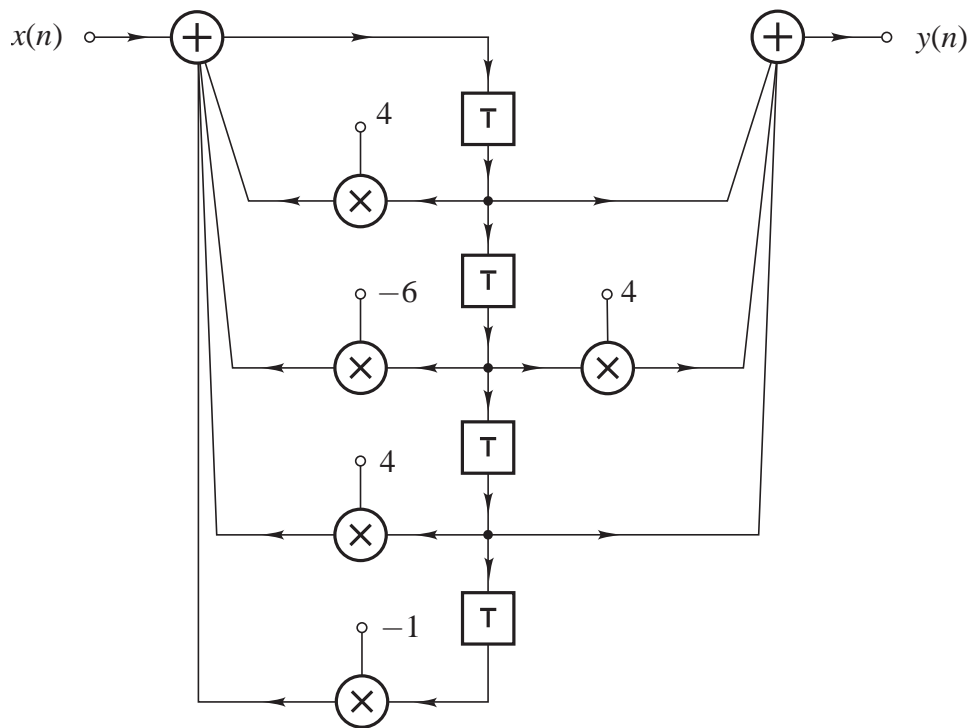
BABBAGE'S BASIC IDEA *Cont'd*

- Assuming that $y(n) = n^3$ is the response of a discrete-time system to a unit impulse, the difference equation

$$y(n] = x(n - 1) + 4x(n - 2) + x(n - 3) + 4y(n - 1) - 6y(n - 2) + 4y(n - 3) - y(n - 4)$$

can be obtained, which represents the discrete-time system shown below.

- In other words, contrary to popular belief, Babbage invented discrete-time systems not computers.



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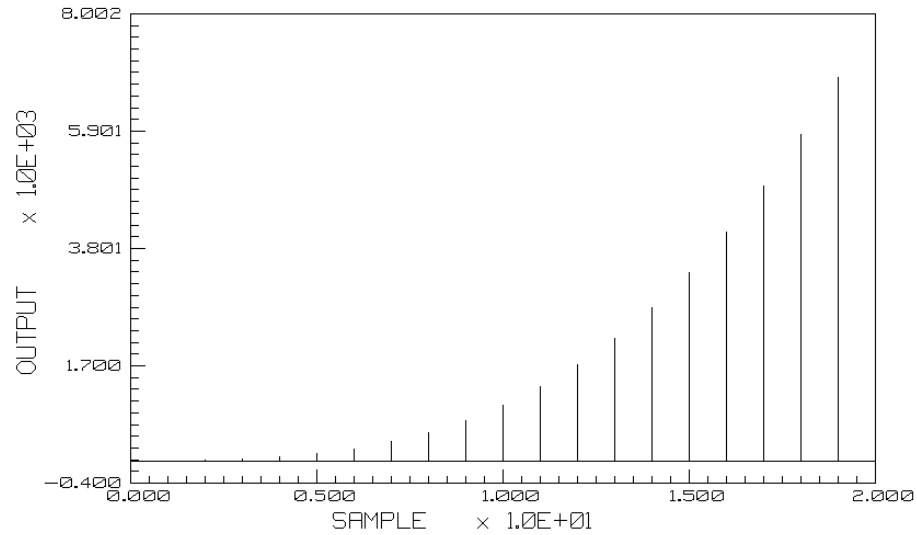
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BABBAGE'S BASIC IDEA *Cont'd*

IMPULSE RESPONSE



NOTE: Actually on the basis of the bounded-input, bounded-output definition of stability, the system is unstable but it will continue generating cubes until it runs into overflow.

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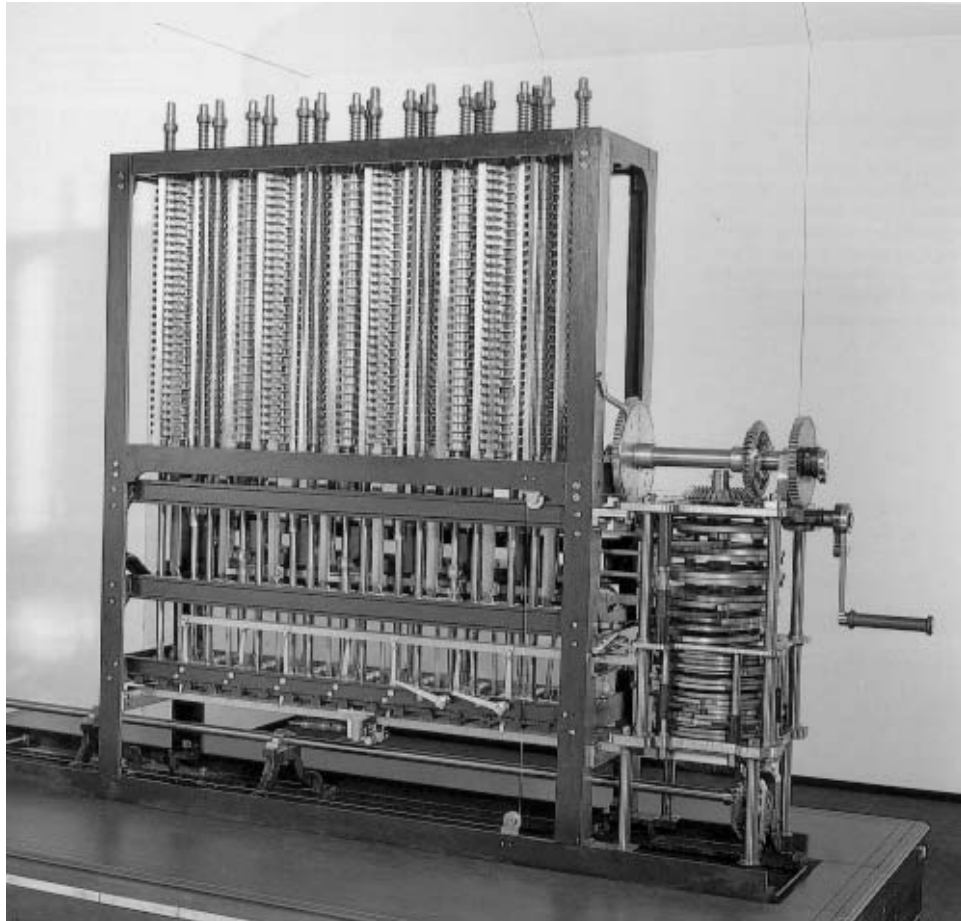
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Babbage's Difference Engine No. 2

- Was designed during 1847-49.
- Was built at the Science Museum, London, U.K., in 1991 (see [3]).
- Measures $2.1 \times 3.4 \times 0.5$ m.
- Weighs 3 tons.
- Can evaluate 7th-order polynomials.
- Was designed to calculate to 30 significant figures.



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CONCLUSIONS

- The underlying ideas of DSP, i.e., sampling and interpolation, have been around for centuries.
- Contrary to popular belief, the discrete-time system has given birth to the computer, not the other way around.

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REFERENCES

1. <http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians>
2. C. B. Boyer, *A History of Mathematics*, Wiley, New York, 1991.
3. D. D. Swade, Redeeming Charles Babbage's Mechanical Computer, *Scientific American*, Feb. 1993.
4. Charles Babbage and his Calculating Engines, Science Museum, London, U.K., 1991.

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